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Finslerian Lie Variations for Dust-Like Matter

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We use the Finslerian Lie variations and obtain the equations of motion of dust-like matter.

KEY WORDS: Finsler; Lie variation; fluids; equations of motion.

1. INTRODUCTION

As is well known (Raigorodski, 1995; Raigorodski *et al.*, 1999; Yano, 1957) the Lie variations in Riemannian geometry are defined, for a geometrical object $A(x)$, by

$$\delta_\xi A = A'(x') - A(x') \tag{1}$$

where we have put the infinitesimal prime transformations in the form

$$x^{i'} = x^i + n^i \delta t \qquad \xi^i = n^i \delta t$$

$$A(x') = A(x) + A_{,l}\xi^l \qquad A_{,l} = \frac{\partial A}{\partial x^l}$$

In (1) $A(x')$ and $A'(x')$ are the components of the object A^n in the coordinate systems x^i and $x^{i'}$, respectively. Therefore we can obtain the following Lie variations:

$$\delta_\xi \varphi = \varphi'(x') - \varphi(x') = \varphi(x) - (\varphi(x) + \varphi_{,l}\xi^l) = -\varphi_{,l}\xi^l$$

$$\delta_\xi A^i = A^{i'}(x') - A^i(x') = \frac{\partial x^{i'}}{\partial x^l} A^l(x) - \left(A^i(x) + A^i_{,l}\xi^l\right)$$

$$= \left(A^i(x) + \xi^i_{,l} A^l\right) - \left(A^i(x) + A^i_{,l}\xi^l\right) = -A^i_{,l}\xi^l + \xi^i_{,l} A^l \tag{2}$$

$$\delta_\xi G_{ik} = -G_{ik,l}\xi^l - G_{lk}\xi^l_{,i} - G_{il}\xi^l_{,k}$$

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etc. Starting from the Riemannian Lie variations (2) we determine the Lie variations in Finsler geometry and obtain the equations of motion of the dust-like matter (incoherent fluid) by means of the variational principle with respect to the Finslerian case.

2. FINSLERIAN LIE VARIATIONS

In this section we define the Finslerian Lie variations. In the case of a vector $A^i(x, \dot{x})$, we can put as in (1)

$$\delta_\xi A^i = A^{i'}(x') - A^i(x') \tag{3}$$

$$x^{i'} = x^i + \xi^i(x), \qquad \dot{x}^{i'} = \dot{x}^i + \xi^i_{,l}\dot{x}^l$$

where in the Finslerian case

$$A^{i'}(x') = \frac{\partial x^{i'}}{\partial x^l}A^l(x) = \frac{\partial(x^i + \xi^i)}{\partial x^l}A^l(x) = A^i(x) + \xi^i_{,l}A^l(x) \tag{4}$$

$$A^i(x') = A^i(x) + A^i_{,l}\xi^l + A^i_{,i}\delta\dot{x}^l$$

$$A^i_{\dot{l}} = \frac{\partial A^i}{\partial \dot{x}^l} \qquad \delta\dot{x}^l = \frac{\partial \xi^l}{\partial x^j}\frac{dx^j}{dt} \tag{5}$$

Therefore we can obtain (Rund, 1959)

$$\delta_\xi A^i = \xi^i_{,l}A^l - A^i_{,l}\xi^l - A^i_{\dot{l}}\xi^l_{,j}\dot{x}^j \tag{6}$$

For the case of a covariant tensor field $G_{ih}(x, \dot{x})$

$$\begin{aligned}\delta_\xi G_{ih} &= G_{i'h'}(x') - G_{ih}(x') \\ &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^h}{\partial x^{h'}}G_{ih} - \left[G_{ih} + G_{ih,l}\xi^l + G_{ih,i}\xi^l_{,m}\dot{x}^m\right] \\ &= \left(G_{ih} - G_{lh}\xi^l_{,i} - G_{il}\xi^l_{,h}\right) - \left(G_{ih} + G_{ih,l}\xi^l + G_{ih,\dot{l}}\xi^l_{,m}\dot{x}^m\right) \\ &= -G_{lh}\xi^l_{,i} - G_{il}\xi^l_{,h} - G_{ih,l}\xi^l - G_{ih,\dot{l}}\xi^l_{,m}\dot{x}^m.\end{aligned}$$

3. EQUATIONS OF MOTION OF THE DUST-LIKE MATTER

First we give the equations of motion for an incoherent fluid in pseudo-Riemannian space using the Lie variation in Raigorodski (1995) and Raigorodski *et al.* (1991), and after that we derive, using the Finslerian Lie variation, the equations of motion of the incoherent fluid in a Finslerian space–time.

The action of a physical system has the from

$$S = \int \alpha\, d\Omega = \int \Lambda\sqrt{-g}\, d\Omega \tag{7}$$

with $\alpha = \Lambda\sqrt{-g}$ where Λ is a scalar formed from the quantities which characterize the examined system (we have in view the generalized coordinates and the velocities), and from the quantities which represent the "background" (the rest).

We shall vary the action S, using the Lie variation.

We have

$$\delta_\xi S = \int \delta_\xi \alpha \, d\Omega = \int [(\delta_\xi \Lambda \sqrt{-g} + \Lambda(\delta_\xi \sqrt{g})] \, d\Omega.$$

Applying the formulas

$$\delta_\xi \Psi = \Psi'(x') - \Psi(x) - \frac{\partial \Psi}{\partial x^i}\xi^i = -\Psi_{,i}\xi^i \tag{8}$$

and

$$\delta_\xi \sqrt{-g} = -\sqrt{-g}\left(\xi^l_{,l} + \frac{1}{\sqrt{-g}} \cdot \frac{\partial\sqrt{-g}}{\partial x^l}\xi^l\right) = -\frac{\partial(\sqrt{-g}\xi^l)}{\partial x^l} \tag{9}$$

we find

$$\delta_\xi S = -\int \Lambda_{,i}\xi^i\sqrt{-g}\, d\Omega - \int \Lambda(\sqrt{-g}\xi^i)_{,i}\, d\Omega.$$

Hence, the Lie variation of the action is always zero,

$$\delta_\xi S = \delta_\xi \int \alpha \, d\Omega = \int \delta_\xi \alpha \, d\Omega = 0, \tag{10}$$

and therefore we should vary either only the quantities related to the examined object, or those which characterize the background.

For this, we rewrite the density of the Lagrangian α in the form

$$\alpha = O_{i\cdots k} \cdot B^{i\cdots k}\sqrt{-g},$$

where $O_{i\cdots k}$ is a tensor which is built from quantities related to the examined system, and the tensor $B^{i\cdots k}$ represents the background.

Having in view (10), we have

$$\int \delta_\xi \alpha \cdot d\Omega = \int (\delta_\xi O_{i\cdots k})B^{i\cdots k}\sqrt{-g}\, d\Omega + \int O_{i\cdots k}\delta_\xi(B^{i\cdots k}\sqrt{-g})\, d\Omega = 0,$$

or

$$\int B^{i\cdots k}\sqrt{-g}(\delta_\xi O_{i\cdots k})\, d\Omega = -\int O_{i\cdots k}\delta_\xi(B^{i\cdots k}\sqrt{-g})\, d\Omega. \tag{11}$$

Therefore, for obtaining the equations which describe the evolution of the physical object (the equations of motion), we have to choose one of two possible ways of applying the principle of least action:

- to submit to Lie variation inside the integral of action only those quantities, which characterize the examined physical object, considering that the characteristics of the background are given (i.e., fixed);
- to vary only the quantities which are related to the background.

This conclusion can be regarded as an interpretation of Mach's principle of relativity, in terms of variational principles (Mach consideres as background the "fixed stars," and as physical object—whose movement is investigated, some "body") (Raigorodski *et al.*, 1999).

We shall examine the continuum of individual particles which do not interact (a dust-like matter, or an incoherent fluid) (Raigorodski, 1995).

The energy–momentum tensor of the incoherent fluid is

$$T^{ik} = \mu u^i u^k, \tag{12}$$

where μ is the invariant density of the mass and

$$u^i = \frac{dx^i}{ds} \tag{13}$$

is the four-dimensional velocity vector in this fluid, which satisfies the continuity equation (the conservation principle in its differential formulation):

$$(\mu u^i \sqrt{-g})_{,i} = 0 \tag{14}$$

Let us imagine that an incoherent fluid moves inside some field, whose potential is a symmetric tensor of rank 2 with components $G_{ik} = G_{ki}$. We assume, that the action can be expressed in the form

$$S = \beta \int G_{ik} \mu u^i u^k \sqrt{-g}\, d\Omega, \tag{15}$$

where β is some constant.

On the basis of the principle of least action, we submit to Lie variation only the field G_{ik}.

We have

$$\delta_\xi S = \beta \int (\delta_\xi G_{ik}) \mu u^i u^k \sqrt{-g}\, d\Omega = 0.$$

Using the relation (2), we obtain

$$\delta S = \beta \left\{ -\int G_{ik,l} \mu u^i u^k \sqrt{-g} \xi^l\, d\Omega - \int G_{lk} \mu u^i u^k \sqrt{-g} \xi^l_{,i}\, d\Omega \right.$$
$$\left. - \int G_{il} \mu u^i u^k \sqrt{-g} \xi^l_{,k}\, d\Omega \right\} = 0. \tag{16}$$

We integrate by parts the last two integrals. Applying the Gauss' theorem and taking into consideration the relation (14) and the fact that on the boundary of the domain of integration the variations of ξ^l vanish, we obtain

$$\delta_\xi S = \alpha \int \left[(G_{lk,i} + G_{il,k} - G_{ik,l}) \mu u^i u^k + G_{ik} \mu u^i u^k_{,l} + G_{il} \mu u^i_{,k} u^k \right] \xi^l \sqrt{-g}\, d\Omega$$

$$= 2\alpha \int \mu \left[G_{il} \frac{du^i}{ds} + \frac{1}{2} (G_{lk,i} + G_{il,k} - G_{ik,l}) u^i u^k \right] \xi^l \sqrt{-g}\, d\Omega = 0. \quad (17)$$

Since ξ^i are arbitrary, we find

$$\mu G_{il} \frac{du^i}{ds} + \mu \frac{1}{2} (G_{lk,i} + G_{il,k} - G_{ik,l}) u^i u^k = 0. \quad (18)$$

We introduce the tensor $G^{\underline{lm}}$, which satisfies the condition

$$G_{il} G^{\underline{lm}} = \delta_i^m, \quad (19)$$

and we multiply this tensor with Eq. (18). We get

$$\frac{du^m}{ds} + \frac{1}{2} G^{\underline{lm}} . (G_{lk,i} + G_{il,k} - G_{ik,l}) u^i u^k = 0. \quad (20)$$

Equations (20) are the equations of motion of an incoherent fluid inside a field whose potentials are the components of the tensor G_{ik}.

It is clear that if this field were missing, then the equations of motion would have been of the form

$$\mu \frac{du^m}{ds} = 0, \quad (21)$$

or,

$$\frac{d^2 x^m}{ds^2} = 0, \quad (22)$$

i.e., the field lines of the incoherent fluid would have been the straight lines

$$x^m = a^{(m)} s + b^{(m)},$$

where $a^{(m)}$ and $b^{(m)}$ are constants.

If the field $G^{\underline{ik}}$ is present, then the noninteracting particles (the dust-like matter) move along curves which are described by Eq. (20). The vector u^m which is tangent to the field-lines varies with time, and determines the direction of the movement of the particles at each point of space–time.

We remark that the potentials G_{ik} of the field, which interacts with the matter, are not determined. The deviation of the movement of the particles from a rectilinear motion is due to the action of the field upon the particles, as follows directly from relation (20). This deviation is governed by the derivatives of the potentials.

The fact that a perfect fluid has pressure p equal to zero, is intrinsically connected with the fact that the movement of the material particles is oriented. Therefore it is reasonable for someone to extend the gravitational source of the energy–momentum tensor T^{ik} in the Finslerian case, for this state of the matter. So we can take that the energy–momentum tensor depends on the position x and the direction $u(x)$. $T^{ik}(x, u(x))$ is defined to be

$$T^{ik}(x, u(x)) = \mu(x)u^i u^k$$

The dependence of T^{ik} by the velocity u extends the relation (12) of an incoherent fluid in the Riemannian space–time.

We shall apply the Finslerian Lie variation to the Finslerian action S of the dust–like matter and obtain the equations of motion.

If the dust-like matter is moving in a field with potentials $G_{ih}(x, \dot{x})$ then the action of it is given by

$$\tilde{S} = a\int \mu G_{ik}u^i u^k \sqrt{-g}\, d\Omega \tag{23}$$

The vector field is tangent to the streamlines of the fluid which satisfies the continuity equation (the conservation law in its differential formulation)

$$(\mu u^i \sqrt{-g})_{,i} = 0$$

In (23), g is the determinant of the Finslerian metric $g_{ik}(x, \dot{x})$ of the four-dimensional space–time and dw is the volume element. Then, using the variational principle $\delta_\xi \tilde{S} = 0$ we can obtain the equation of the dust-like matter as follows:

$$G_{li}\frac{du^i}{ds} + \frac{1}{2}(G_{lh,i} + G_{il,h} - G_{ih,l})u^i u^k + \frac{1}{2}(G_{ih,\dot{l}})_{,m}\dot{x}^m u^i u^h = 0 \tag{24}$$

Concerning the term $(G_{ih,\dot{l}})_{,m}$, in particular the term $G_{ih,\dot{l}}$ its physical meaning is a *direction-dependent force* which causes the deviation from the Riemannian equation of motion or the Riemannian geodesics. Since $G_{ik,\dot{l}}$ gives the Cartan's vertical connection coefficients C_{ihl} with respect to the Finslerian G_{ih} field (Rund, 1959), the gravity $C_{ihl,m}$ contained in the term $(G_{ih,\dot{l}})_{,m}\dot{x}^m \approx C_{ihl,m}\dot{x}^m$ can be tensorially absorbed into the curvature P_{iklh}. The definition of P_{iklh} is given by

$$F^{-1}P_{jikh} = C_{kih,j} - C_{jkh,i} - C_{kim}C^m_{jh,r}y^r + C^m_{jk}C_{mih,r}y^r$$

This fact is very interesting and important because the curvature P_{iklh} has not fully been considered in physical and geometrical problems until now. The curvature P_{iklm} governs the intermediate state between the x-dependent state, represented by the curvature R_{ilhm} and the $\dot{x}$-dependent state, represented by the curvature S_{ilhm} (as to the definition of R_{ilhm} and S_{ilhm} see (Miron and Anastasiei, 1997; Rund, 1959).

The equation of motion (24) can be written

$$\frac{du^m}{ds} + \frac{1}{2}\tilde{\gamma}^m_{il}(x, \dot{x})u^i u^l + \frac{1}{2}G^{\underline{lm}}(G_{ih,i})_{,k}\dot{x}^k u^i u^h = 0 \tag{25}$$

where we introduced the tensor $G^{\underline{lm}}$ which satisfies the condition $G^{\underline{lm}}G_{il} = \delta^m_i$. The equations of motion for the dust-like matter in the Finslerian concept take the final form

$$\frac{\delta u^m}{\delta s} + C_{ihl,k}G^{\underline{lm}}\dot{x}^k u^i u^h = 0 \tag{26}$$

where we have put

$$\frac{\delta u^m}{\delta s} = \frac{du^m}{ds} + \tilde{\gamma}^m_{il}(x, \dot{x})u^i u^l$$

where $\tilde{\gamma}^m_{il}(x, \dot{x})$ represents the Christoffel symbols of Finslerian space–time.

Remark 3.1. In the case that the Finslerian space–time is of Berwald type, $C_{ihl,k} = 0$, $\tilde{\gamma}^m_{il} = \tilde{\gamma}^m_{il}(x)$, the equations of motion (26) are reduced to the Riemannian ones

$$\frac{\delta u^m}{\delta s} = \frac{du^m}{ds} + \tilde{\gamma}^m_{il}(x)u^i u^l \tag{27}$$

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